Trying To Subjugate The Infinite

Adrian Moore

The infinite is standardly conceived as that which is endless, unlimited, unsurveyable, immeasurable. There are alternative conceptions. But let us, for the purposes of this essay, confine attention to this standard conception.

From the time of the early Greeks, the infinite, so conceived, has aroused suspicion. This suspicion has been due partly to the fact that we can never encounter the infinite in experience, and partly to the fact that the concept seems to be riddled with paradoxes. Thus there is the famous paradox of Achilles and the tortoise, formulated by Zeno of Elea.[1] In this paradox, Achilles, who runs much faster than the tortoise, lets it start a certain distance ahead of him in a race. The paradox is that Achilles seems never to be able to overtake the tortoise, no matter how great the difference between their speeds. For in order to do so, he must first reach the point at which the tortoise starts, by which time the tortoise will have advanced a fraction of the distance initially separating them; he must then make up this new distance, by which time the tortoise will have advanced again; he must then make up *this* new distance, by which time the tortoise will have advanced yet again; and so on *ad infinitum*. A closely related paradox, likewise formulated by Zeno, is that Achilles can never get from one end of the racecourse to the other. For in order to do so, he must first reach the midway point; he must then reach the three-quarters point; he must then reach the seven-eighths point; he must then reach the fifteen-sixteenths point; and so on *ad infinitum*.

There is also a family of paradoxes, known since medieval times, based on the principle that if it is possible to pair off all the members of one set with all the members of another, then the two sets must have exactly the same number of members. (For instance, in a non-polygamous society, there must be just as many husbands as there are wives.) This principle looks incontestable. But when it is applied to infinite sets, it seems to flout Euclid’s notion that the whole is greater than the part. For example, it is possible to pair off all the positive integers with all those that are even. We can pair off 1 with 2, 2 with 4, 3 with 6, 4 with 8, and so on. (The mathematician Hilbert, who used to lecture on the infinite, gave vivid expression to such paradoxes by inviting his audience to imagine a hotel with infinitely many rooms. He supposed that one night, when the hotel was fully occupied, a traveller turned up, seeking a room for the night. Despite the fact that the hotel was fully occupied, a room could be found for this traveller. For the person in Room #1 could move into Room #2, the person in Room #2 could move into Room #3, the person in Room #3 could move into Room #4, the person in Room #4 could move into Room #5, and so on. This would release Room #1 for the traveller, without depriving anyone else of a room. Indeed, even if infinitely many travellers turned up, each seeking a room for the night, rooms could be found for them. For the person in Room #1 could move into Room #2, the person in Room #2 could move into Room #4, the person in Room #3 could move into Room #6, the person in Room #4 could move into Room #8, and so on. This would release the infinitely many odd-numbered rooms for all the newcomers, without depriving anyone else of a room.)

Aristotle, well aware of the problems that afflict the concept of the infinite, and yet reluctant to eschew the concept completely, famously responded to this dilemma by drawing a distinction between what he called 'the actual infinite' and 'the potential infinite'. The actual infinite is that whose infinitude exists, or is given, at some point *in* time. The potential infinite is that whose infinitude exists, or is given, *over* time. Thus imagine a clock endlessly ticking. Its ticking is potentially, but never actually, infinite. Now, all the objections to the infinite, Aristotle insisted, are objections to the actual infinite. They are objections to the idea of an infinitude which is given all at once. The potential infinite, by contrast, is a fundamental feature of reality. It is there to be acknowledged in any process that can never end: for example, in the process of counting; or in various processes of division; or in the passage of time itself. Paradoxes such as Zeno's arise because we fail to pay due heed to this distinction. Having seen, for instance, that there can be no end to the process of dividing the racecourse, we somehow imagine that all those possible future divisions are somehow already in effect there. We come to view the racecourse as already divided into infinitely many parts, and it is easy then for such paradoxes to take hold.[2]

Aristotle's view proved enormously influential: its importance, for subsequent discussion of the infinite, is hard to exaggerate. For well over two thousand years it more or less had the status of orthodoxy. But later thinkers, unlike Aristotle himself, tended to take the references to time in the actual/potential distinction as a metaphor for something more abstract. Existing 'in time', or existing 'all at once', came to assume broader meanings than they had for Aristotle. Eventually, exception to the actual infinite became exception to the very idea that the infinite could be a legitimate object of mathematical study.

The received wisdom nowadays is that this orthodoxy was finally overturned in the nineteenth century, when Cantor presented a coherent, rigorous, systematic mathematical theory of the infinite. Cantor took the paradoxes in his stride, formulated precisely what is going on in them, and then incorporated these formulations into his theory. No longer, it seemed, did the (actual) infinite have to be treated with mistrust and hostility.

In due course I shall query whether the situation is as simple as this standard account suggests. But first I want to sketch some of the most notable features of Cantor's theory.

Cantor accepted that there are as many even positive integers as there are positive integers altogether. He did not flinch at the idea that the part can be as great as the whole. (Indeed we can use this idea to define the infinite, at least in its applications to sets. A set is infinite if it is no greater than one of its parts. More precisely, a set is infinite if it has just as many members as one of its proper subsets.) Cantor did not, however, go to the other extreme of urging that all infinite sets are the same size (a conclusion which, in its own way, would not have been all that repugnant to commonsense). On the contrary, much of the revolutionary impact of his work came in his demonstration that, even when conceived in these terms, not all infinite sets are the same size. This is a consequence of what is known as Cantor's theorem: no set, and in particular no infinite set, has as many members as it has subsets. In other words, no set is as big as the set of its subsets.

To see why Cantor's theorem holds, let us consider its application to the set of positive integers. Suppose we pair off individual positive integers with sets of positive integers. For instance, we might begin by pairing off 1 with the set of odd positive integers {1, 3, 5, 7, …}, 2 with the set of even positive integers {2, 4, 6, 8, …}, 3 with the set of square positive integers {1, 4, 9, 16, …}, and 4 with the set of prime positive integers {2, 3, 5, 7, …}. The point is: however we begin, and however we proceed, there is guaranteed to be at least one set of positive integers that is left out. That is, there is guaranteed to be at least one set of positive integers that is not paired off with any individual positive integer. Why? Well, consider the fact that some individual positive integers will belong to the sets with which they are paired off and others will not. Call positive integers of the latter kind *diagonal*. (I have used this term because the proof technique at work here is often called 'diagonalisation'.) In the example above, neither 1 nor 2 is diagonal, for each of them *does* belong to the set with which it is paired off; but both 3 and 4 *are* diagonal, for each of them does *not* belong to the set with which it is paired off. Now consider the set of diagonal positive integers. Call this set D.

Question: With which positive integer, if any, is D paired off? *Answer:* None. For let us suppose that it *is* paired off with some positive integer—call it *d*—and let us consider whether *d* is diagonal or not. By definition, *d* (like any other positive integer) is diagonal if and only if it does not belong to the set with which it is paired off. But since the set with which *d* is paired off is *D*—the set of diagonal positive integers—this is tantamount to saying that d is diagonal if and only if it is not diagonal, which is a blatant contradiction. So however we pair off individual positive integers with sets of positive integers, the set of diagonal positive integers that we thereby create is guaranteed to be left out. Hence there are more sets of positive integers than there are individual positive integers.

Cantor went on to devise infinite cardinals: numbers that can be used to measure the sizes of infinite sets. He invented a kind of arithmetic for them as well. Having first suitably defined his terms, he explored what happens when one infinite cardinal is added to another, or multiplied by another, or raised to the power of another. His work showed mathematical craftsmanship of the very highest calibre.[3]

But he needed to proceed cautiously. His work made indispensable use of the idea of a set (as glimpsed above). But what is a set? On one very natural conception of what a set is, often referred to as the 'naïve' conception, a set is something that collects together all those things that have a given property. And for *any* given property, there is, on the naïve conception, a set corresponding to it. Thus corresponding to the property of being a planet, there is the set of planets; corresponding to the property of being a person, there is the set of people; corresponding to the property of being a set, there is the set of sets; corresponding to the property of being a square positive integer, there is the set of square positive integers; and so forth. However, the naïve conception can be shown to be incoherent. For suppose there is a set corresponding to any given property. And consider the fact—here I am still presupposing the naïve conception—that some sets belong to themselves, and some do not. Thus the set of sets is of the former kind, because it is itself a set. But the set of planets is of the latter kind, because it is not itself a planet. Now consider the set corresponding to the property of being a set that does *not* belong to itself. In other words, consider the sets of sets that do not belong to themselves. Call this set *R. Question:* Does *R* belong to itself or not? *Answer:* By definition, *R* (like any other set) belongs to *R* if and only if it does not belong to itself. But this is a blatant contradiction. So the naïve conception is incoherent.

(This, incidentally, is Russell's famous paradox. There is a striking resemblance between the reasoning involved in this paradox and the reasoning involved in Cantor's theorem. The connections between the two are very deep. Indeed it was by reflecting on Cantor's theorem that Russell first stumbled across his paradox.[4])

In order to safeguard his theory from this kind of incoherence, Cantor needed to operate with a somewhat more sophisticated conception of a set. The conception with which he operated is often referred to as the 'iterative' conception. On the iterative conception, a set is something whose existence is parasitic on that of its members: the members exist 'first'. Thus there are, to begin with, all those things which are not sets (planets, people, positive integers, and so forth). Then there are sets of these things. Then there are sets of *these* things. And so on, without end. Each thing, and in particular each set, belongs to countless further sets. But there never comes a set to which every set belongs. There is no set of all sets. How does this escape the incoherence in the naïve conception? Well, on the iterative conception, no set belongs to itself. Hence *R*, if it existed, would be the set of all sets. But, to repeat, there is no set of all sets. There is no such thing as *R*.

I described the naïve conception above as 'very natural'. But there is something quite natural about the iterative conception too. The iterative conception has great intuitive appeal. But is it not also strikingly Aristotelian? Notice the temporal metaphor that sustains it. Sets are depicted as coming into being 'after' their members, in such a way that there are always more to come. Their collective infinitude, as opposed to the infinitude of any one of them, is potential, not actual. Moreover, it is this collective infinitude that has best claim to the title. For the properties that I listed at the outset as characterising the standard conception of the infinite— endlessness, unlimitedness, unsurveyability, immeasurability— more properly apply to the entire hierarchy than to anything in it. This is partly because of the very success that Cantor enjoyed in subjecting the sets *in* the hierarchy to careful mathematical scrutiny. For example, he showed that the set of positive integers is limited in size. (The set of sets of positive integers has more members.) He also showed that we can give a precise mathematical measure to how big it is. There is a sense, then, given that limitedness, measurability, and the like are part of the standard conception of the finite, in which he established that the set of positive integers is 'really' finite, and that what is 'really' infinite is something of an altogether kind. (He was not himself averse to talking in these terms.) In a curious and ironical kind of way, his work served, in the end, to corroborate the Aristotelian orthodoxy that 'real' infinitude can never be actual.

This is a view that I have defended elsewhere.[5] A number of mathematicians and philosophers have objected to my idea that, on Cantor's showing, the set of positive integers is 'really' finite. They complain that this idea is not only at variance with standard mathematical terminology, but also, contrary to what I seem to be suggesting, with what most people would say.

Well, certainly most people would say that the set of positive integers is 'really' infinite. But then again, most people are unaware of Cantor's achievements. They would also deny that the set of positive integers has a precise infinite size, strictly smaller than that of the set of sets of positive integers. My point is not about what most people would say. It is about how they understand their terms; and about how that understanding is best able, for any given purpose, to absorb the shock of Cantor's results. Nothing here is forced on us. We could say that some infinite sets are bigger than others. We could say that the set of positive integers is only finite. We could hold back from saying either and deny that the set of positive integers exists. (After all, it is an integral part of the iterative conception to deny that every property has a set corresponding to it.)

If the task at hand is to articulate certain standard mathematical results, then I would not advocate using anything other than standard mathematical terminology. But I would urge mathematicians and philosophers to exercise more caution than usual when it comes to interpreting these results, and in particular when it comes to saying how they bear on traditional conceptions of the infinite. The truly infinite, I suggest, cannot be subjugated.

A.W. Moore
St Hugh's College Oxford
adrian.moore@philosophy.oxford.ac.uk

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[1] See Aristotle, *Physics*, Bk VI, Ch. 9, in his *The Complete Works*, ed. Jonathan Barnes (Princeton: Princeton University Press, 1984).
[2] See e.g. Aristotle, *Physics*, Bk III, Chs 6 – 7, Bk VI, Ch. 9, and Bk VIII, Ch. 8, in *ibid*.
[3] See e.g. Georg Cantor, *Contributions to the Founding of the Theory of Transfinite Numbers*, trans. Philip E.B. Jourdain (New York: Dover, 1955).
[4] See Bertrand Russell, *Introduction to Mathematical Philosophy* (London: Allen & Unwin Ltd, 1919), pp. 135 – 136.
[5] A.W. Moore, *The Infinite*, 2nd edn (London: Routledge, 2001).