Numbers and Ideas

John Burgess

I

Philosophy is a subject in which there is very little agreement. This is so almost by definition, for if it happens that in some area of philosophy inquirers begin to achieve stable agreement about some substantial range of issues, straight-away one ceases to think of that area as part of 'philosophy', and begins to call it something else. This happened with physics or 'natural philosophy' in the seventeenth century, and has happened with any number of other disciplines in the centuries since. Philosophy is left with whatever remains a matter of doubt and dispute.

Philosophy of mathematics, in particular, is an area where there are very profound disagreements. In this respect philosophy of mathematics is radically unlike mathematics itself, where there are today scarcely ever any controversies over the correctness of important results, once published in refereed journals. Some professional mathematicians are also amateur philosophers, and the best way for an observer to guess whether such persons are talking mathematics or philosophy on a given occasion is to look whether they are agreeing or disagreeing.

One major issue dividing philosophers of mathematics is that of the nature and existence of mathematical objects and entities, such as *numbers*, by which I will always mean *positive integers* 1, 2, 3, and so on. The problem arises because, though it is common to contrast matter and mind as if the two exhausted the possibilities, numbers do not fit comfortably into either the material or the mental category.

Clearly numbers are not material bodies. The so-called numbers on the front of a house, marking its street address, may indeed be made of brass or wood or plastic. But these 'numbers' are not the numbers we speak of when we say that two is an even number, or that three is an odd number, or that both are prime numbers. Rather, they are *numerals*, or names of numbers.

Almost equally clearly, numbers are not mental in the way that, say, dreams or headaches are. They are not private to an individual. One does not speak of my number two and your number two, his number two and her number two, but simply of *the* number two. The individual, say a school child doing a simple sum, experiences the numbers as something external, about which he or she is *not* free to think whatever he or she wants.

But if numbers are not material bodies or private experiences, what (if anything) are they? Among professional academic philosophers the most commonly held views are two, for want of better terms called *realism* and *nominalism*.

Realism maintains that numbers exist, and are of a very different nature from human ideas: indeed, they differ quite as much from human ideas as they do from material bodies. They are *abstract* entities, to which it makes no sense to ascribe a position in space or date in time, and which are not causally active or acted upon. There is nowhere to go to look for a number, and you can't do anything to a number, any more than a number can do anything to you.

Nominalism maintains that numbers do not exist, and that theorems of mathematics asserting the existence of numbers are untrue, just like fairy tales asserting the existence of gnomes. To be sure, much of mathematics is applicable in science and everyday life in a way that fairy tales generally are not, but that, according to nominalists, only shows it is a *useful* fiction, not that it is non-fiction.

There are problems for both opposing philosophical views, and the problems of each are cited by the adherents of the other as reasons for embracing *it* instead. And formerly there were among philosophers also many who maintained a third view, *conceptualism* or *idealism*, according to which numbers exist, but only as shared human concepts or ideas.

The view has traditionally been popular among anthropologists and other social scientists, whose special subject matter is precisely the shared ideas of a culture. They point out that taking numbers to be such shared or *communal* ideas sufficiently explains why the school child doing a simple sum does not feel free to make up an answer at will. If numbers are ideas shared by a culture, no one member of that culture has the authority to change the rules of addition, any more than to change the rules of grammar of the culture's language.

The anthropological view has also found adherents among mathematics educators. Rather more surprisingly, the same view has won adherents among the minority of professional mathematicians who are also amateur philosophers.[1]

Conceptualist and idealist views, however, were subjected along with other nineteenth century views to a scathing critique by the late nineteenth century German mathematician and philosopher Gottlob Frege.[2] Largely as a result of that critique, the anthropological view today has virtually no adherents among professional academic philosophers. Its rejection is one of the rare cases of general agreement and consensus on an issue in philosophy.

Precisely because there is such general agreement, philosophers seldom stop to explain, in language more modern than Frege's, just what is wrong with the view that so many anthropologists, sociologists, psychologists, mathematics educators, and even mathematicians have found attractive. It is this task of explanation that I will be undertaking in the present essay, using an example of a kind that definitely would not have been used by Frege.

II

Let us begin by considering the proposition that *Bigfoot*, also known as the *Sasquatch* — a cousin of the *Abominable Snowman* or *Yeti* — exists in the realm of shared human ideas and concepts. Now certainly there is *something* in the neighbourhood that exists in the realm of shared human ideas and concepts, namely, the shared human idea or concept of Bigfoot. This is the idea of a large, hairy, humanoid creature inhabiting the wilder parts of the Pacific Northwest, from Northern California to British Columbia.

There are even people who claim to have sighted individual Bigfeet, and have formed ideas of these individuals, even to the point of giving them names like ‘Harry’ or ‘Harriet’. The idea of an individual Bigfoot includes the traits that are common to all Bigfeet according to the general idea of Bigfoot, but also more specific elements: for instance, Harry is male and Harriet is female. These ideas of individual Bigfeet are less widely shared than the idea of the species, but we may suppose they are at least shared among members of the International Society for Cryptozoology, who take a special interest in such things.

The majority view among zoologists is that there do not, in fact, exist any large, hairy, humanoid creatures, and that the alleged sightings of Harry, Harriet, and other individual Bigfeet were either illusions or hoaxes. But I ask you to join me in assuming, just for the moment, that majority is wrong, and that creatures of the kind indicated, including Harry and Harriet, do exist. On this assumption, I will argue, two things should be clear. The first is that Harry, Harriet, and other large, hairy, humanoid creatures inhabiting the wilder parts of the Pacific Northwest are very different sorts of things from shared human ideas and concepts, and in particular, are very different sorts of things from the ideas and concepts of Harry, of Harriet, and of Bigfoot in general. They differ in absolutely fundamental respects, for instance, in their location in space and time.

Let us consider space, for instance. (Similar considerations would apply to time.) It is not clear whether or where a shared human idea or concept should be thought of as located in space, but presumably if it is located anywhere, it is located where the human beings who share it are located. Thus if the International Society for Cryptozoology holds its annual convention on the banks of Loch Ness, the idea of Bigfoot in general, and the ideas of Harry and Harriet in particular, are located mainly in Scotland. Harry, Harriet, and the rest of their kind, however, are still located in Washington or Oregon or thereabouts. The creatures cannot be the ideas, because the two are located in different places.

The creatures differ from the ideas also in respect of how many of them there are. People have ideas of Harry, Harriet, and several more Bigfeet that have allegedly come into contact with human beings; but there are supposed to be, according to the minority view I have asked you to assume for the moment, more Bigfeet than just these: more individuals like Harry and Harriet than there are shared human ideas of individual Bigfeet. So again the creatures cannot be the ideas, since there are more of the former than of the latter.

A second point I hope will be clear is that it is the flesh-and-blood creatures, not the ideas, that are the Bigfeet. The term ‘Bigfoot’ refers to the inhabitants of the wilds of Washington and Oregon, not to the contents of minds or brains of the crypto-zoologists assembled in Scotland. If we wish to refer to the latter, we must use some other expression than the word ‘Bigfoot’, such as the phrase ‘the idea of Bigfoot’. In short, on the minority view, according to which the flesh-and-blood creatures do exist, the following is the case: Bigfeet, being flesh-and-blood creatures, are not ideas, and are more numerous than the ideas of them and located in a different place from those ideas.

III

Are things any different on the majority view? It is when one assumes that there are no such flesh-and-blood creatures that some are tempted to say that the Bigfoot in general, or Harry and Harriet in particular, are human ideas. I think this temptation should be rejected.

Let me say straight-away that it would be pointless to object to someone expressing *disbelief* in Bigfoot by saying, 'Bigfoot exists only in the imagination of the credulous,' or something of the sort. Someone might well say this — I might well say it myself, for that matter, when not talking philosophy — and mean it only as a manner of speaking, as a way of saying, 'Bigfoot doesn't exist at all, though some credulous persons imagine that it does.' The proposition I want to consider, however, is that Bigfoot *literally* does exist, but only in the realm of shared human ideas and concepts, where, according to the anthropological view, numbers also have their being.

To indicate the reasons why I reject this proposition, suppose the population of some endangered forest or swamp species falls until there is only one left. So long as this one surviving flesh-and-blood or wood-and-sap organism lives, considerations of the kind already adduced in the case of Bigfoot indicate that *it* is the only member of the species, and *it* is not an idea, from which it follows that the members of the species are not ideas.

Now suppose this last survivor also perishes. Are we now to say that the species still has members, but that the members of the species are now ideas? Should we say that the species has not become extinct but rather has undergone a *metamorphosis*, transcending its former carnal or xyline nature, and taken on a conceptual essence: that its members have cast aside their fleshly or wooden bodies, and are now made of whatever ideas are made of? Should we say that the species has undertaken a *migration*, abandoning the woods or marshes that were once its home, and occupying now instead a niche in the minds or brains of human subjects?

It seems to me about as plain as anything can be in philosophy — where admittedly things are never as plain as they are in some other disciplines — that this is *not* what we should say, and that the *correct* way to the describe the situation is by saying that creatures of this animal or plant species *simply no longer exist at all*, though of course human ideas about them do exist, and may perhaps continue to exist as long as the human species does.

Likewise in the case of Bigfoot. If the forest creature exists, then Bigfoot is that forest creature, and is something very different from an idea. If the forest creature does not exist, then Bigfoot is, so to speak, even *more* different from an idea: for in that case Bigfoot is *nothing*, while the idea is at least *something*, and what could be more different than something and nothing?

The case is the same, I maintain, with our shared human ideas and concept of number in general, and of individual numbers such as one or two or three. (Again the individual ideas contain whatever is contained in the general idea, plus additional distinguishing elements. We no longer imagine, as did the Pythagoreans, that two is female and three is male, but, for instance, two is even and three is odd.) These ideas

are clear enough, I maintain, to indicate that one, two, three, and the other numbers, if they exist at all, do not have the same sort of spatial or temporal features as human ideas, and above all are more numerous than human ideas could possibly be.

Taking first issues of time and place, mathematics is used throughout science, and mathematical objects and entities are referred to in all its branches, including those like cosmology that deal with times and places very remote from any inhabited by human beings. Are we to say that a cosmologist's estimates of the relative numbers of heavy and light elements at a certain stage in the early evolution of the universe must be wrong, because there were no numbers at all back then, no human beings having yet evolved to create them? Surely not.

And then there is the matter of infinity. It is a crucial feature of the concept of the number system that it has infinitely many elements, that there are infinitely many numbers. But surely human beings have formed ideas or concepts of only finitely many of them. There simply are not enough human ideas and concepts for each number to be one. *Some* numbers at least must therefore either enjoy a mode of existence different from that of any human idea, as realists maintain, or else must simply fail to exist, as nominalists hold. And is it not preposterous to maintain that while one of the pair realism or nominalism gives the correct account of mathematical existence in the case of *some* numbers, conceptualism is correct *for the rest*? Surely the question of the existence and nature of numbers has a *uniform* answer, and if conceptualism fails in *any* case, then it must fail in *all*.

Such then, are some of the principal reasons why I and almost all professional philosophers of mathematics reject conceptualism, and consider the only real issue to be that between nominalism and realism. This last issue is far too large to be thrashed out here, but I do wish to say a word about it, and in particular about the character of the *realist* position, which very often tends to be misrepresented. Nominalists do not believe in numbers because they cannot *see* them (or see any visible effects *caused* by them), and tend to represent their opponents as claiming that they *can* see them.

According to an old story, Plato was once lecturing in his Academy on his Forms, and was speaking of the Forms of 'tableness' and 'cupness'. Diogenes the Cynic interrupted and said, 'O Plato, I see the table and the cup, but the tableness and the cupness I do not see.' To this Plato replied, 'Very naturally, Diogenes, since you have eyes, by which material things are perceived, but lack Intellect, by which the Forms are seen.'[3]

Nominalists tend to represent their opponents as Platonists, maintaining that if numbers do not emit electromagnetic radiation to which the eye is sensitive, then they must be emitting something else, perhaps *noetic rays*, which can be sensed by some other organ, perhaps the *pineal gland*. This, however, is a misrepresentation of realism. Or at least, I have never known a single realist who was in any meaningful sense a Platonist.

What is actually the case is that anti-nominalists take much more seriously than nominalists the thought that *mathematics* is a human creation, since mathematics is a body of theory expressed in language, and *language* is a human creation.

Now creating a language involves creating certain rules for its use. Among these is, I believe, a rule to the effect that tense and date are not to be applied to mathematical existence assertions. One can say 'There exist infinitely many prime numbers,' but to ask 'How many of them already existed in 1000 BCE, or during the Cenozoic Era?' is to commit a kind of grammatical solecism.

Nominalists say they are opposed to the view that numbers are 'eternal', existing 'outside of time'. But to say that numbers are 'eternal' is a misleadingly Platonistic way of putting the simple negative grammatical fact of the inapplicability of tense distinctions in mathematical contexts. That simple grammatical point is all the realist really believes about the 'timelessness' of number.

(By contrast with the case of the numbers themselves, it makes perfect sense to ask whether the *idea* or *concept* of prime number had emerged by 1000 BCE — the issue involved would be that of the interpretation of certain Babylonian tablets and Egyptian papyri — and it makes perfect sense to assert that it had *not* emerged in the Age of the Dinosaurs. This difference between the 'timeless' numbers proper and datable ideas of them was one of the points I was arguing in rejecting conceptualism.)

Likewise, there are certain rules or standards as to what counts as adequate or sufficient to establish or prove a mathematical existence theorem, and by these rules Euclid's Theorem on the existence of infinitely many prime numbers is as well-established as anything can be.

The *nominalists* assume that they have an understanding of what it would be for a mathematical object or entity to exist that is independent of ordinary mathematical standards of sufficient proof, by reference to which understanding they can *criticize* the ordinary mathematical standards. So-called realism is really just *skepticism* about the existence of any understanding of what 'existence' means in mathematics that is independent of ordinary mathematical standards for evaluating existence proofs. The nominalist denies the existence of numbers, while the realist denies that the nominalist understands what is meant by 'existence' as applied to numbers.

Thus the realists think the nominalists are confused. But realists and nominalists agree that the conceptualists are confused, and while I cannot hope to have convinced anyone by the foregoing very brief remarks that the realists are right as against the nominalists, I hope I have convinced some of you that realists and nominalists are right in their common opposition to conceptualism.

John P. Burgess
Princeton University

[1] The classical expression of the anthropological view is that of Leslie A. White, 'The locus of mathematical reality: an anthropological footnote', *Philosophy of Science* 14 (1947), 289-303, more readily available in Newman, J. R. (ed.) *The World of Mathematics* vol. 4 (New York: Simon & Schuster, 1956). For a recent endorsement by the mathematician Reuben Hersh, see his *What is Mathematics, Really?* (Oxford: Oxford University Press, 1997), a book that makes a professional philosopher's hair stand on end.

[2] *Grundlagen der Arithmetik* (1884), translated by J. L. Austin as *The Foundations of Arithmetic*, (London: Blackwell, 1950), reprinted in paperback (New York: Harper & Row, 1960); the critical portions (the part of the book relevant to the present essay) are reprinted in Benacerraf, P. & Putnam, H. (eds.) *Philosophy of Mathematics: Selected Readings* 2nd ed. (Cambridge: Cambridge University Press, 1983).

[3] See Diogenes Laertius, 'Diogenes [the Cynic]', in *Lives and Opinions of Eminent Philosophers*, translated by R. D. Hicks, Loeb Classical Library (Cambridge: Harvard University Press, 1925).